

QUANTITATIVE BOUNDS IN THE POLYNOMIAL SZEMERÉDI THEOREM: THE HOMOGENEOUS CASE

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ABSTRACT. We obtain quantitative bounds in the polynomial Szemerédi theorem of Bergelson and Leibman, provided the polynomials are homogeneous and of the same degree. Such configurations include arithmetic progressions with common difference equal to a perfect k th power.

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1. INTRODUCTION

On a number of occasions Gowers [Gow98, Gow00, Gow01] has asked for an alternative proof of the polynomial Szemerédi theorem of Bergelson and Leibman [BL96], in particular a proof yielding quantitative bounds. The purpose of this paper is to provide a special case of such a result.

Theorem 1.1. *Let $c_1, \dots, c_n \in \mathbb{Z}$. If $A \subset [N] := \{1, 2, \dots, N\}$ lacks configurations of the form*

$$x, \quad x + c_1 y^k, \quad \dots, \quad x + c_n y^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\} \quad (1.1)$$

then A satisfies the size bound

$$|A| \ll_{\mathbf{c},k} N(\log \log N)^{-c(n,k)}. \quad (1.2)$$

Here $c(n, k)$ is a positive absolute constant dependent only on the length and degree of the configuration (1.1).

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In comparison, the original Bergelson–Leibman theorem states that for polynomials $P_1, \dots, P_n \in \mathbb{Z}[x]$ with zero constant term, a set $A \subset [N]$ lacking configurations of the form

$$x, x + P_1(y), \dots, x + P_n(y) \quad \text{with } y \in \mathbb{Z} \setminus \{0\} \quad (1.3)$$

satisfies the size bound $|A| = o_{\mathbf{P}}(N)$.

Theorem 1.1 has a seemingly more general consequence.

Corollary 1.2. *Let $P_1, \dots, P_n \in \mathbb{Z}[y_1, \dots, y_m]$ be homogeneous polynomials, all of degree k , and let K denote a finite union of proper subspaces of \mathbb{R}^m . If $A \subset [N]$ lacks configurations of the form*

$$x, x + P_1(\mathbf{y}), \dots, x + P_n(\mathbf{y}) \quad \text{with } \mathbf{y} \in \mathbb{Z}^m \setminus K \quad (1.4)$$

then A satisfies the size bound

$$|A| \ll_{\mathbf{P}, K} N(\log \log N)^{-c(n, k)}.$$

Theorem 1.1 also yields the first reasonable bounds for certain cases of the polynomial van der Waerden theorem.

Definition (Polynomial van der Waerden number). Given integer polynomials $P_1, \dots, P_n \in \mathbb{Z}[y]$ define the *van der Waerden number* $W(r, \mathbf{P})$ to be the least positive integer N such that any r -colouring of $[N]$ results in a monochromatic configuration of the form (1.3).

When all P_i have zero constant term, the existence of $W(r, \mathbf{P})$ is a consequence of the Bergelson–Leibman theorem. Walters [Wal00] has given a colour-focusing argument which yields (at best) Ackermann-type bounds for $W(r, \mathbf{P})$. The following result combines Theorem 1.1 with the observation that any r -colouring of $[N]$ gives a colour class of size at least N/r .

Corollary 1.3. *If $P_i = c_i y^k$ for $i = 1, \dots, n$ then there exist constants $C_1 = C_1(\mathbf{P})$ and $C_2 = C_2(n, k)$ such that*

$$W(r, \mathbf{P}) \leq \exp \exp (C_1 r^{C_2})$$

Previous results of the type recorded in Theorem 1.1 either concern linear configurations (when $k = 1$) or two-point non-linear configurations (when $n = 1$). For the linear case, the first bounds of the form (1.2) were obtained by Roth [Rot53] when $n = 2$, and by Gowers [Gow01] when $n \geq 3$. It is this latter approach we generalise. Gowers in fact provides the estimate

$$c(n, 1) \geq 2^{-2^{n+9}}, \quad (1.5)$$

something we are not able to replicate when $k > 1$ (see §2 for more on this). For linear configurations with $n = 2$ or 3 there are further improved bounds available, see [Blo14] or [GT09] respectively.

For two-point non-linear configurations, the first quantitative bounds were obtained by Sárközy [Sá78a, Sá78b] and the most general type of configuration were tackled by Lucier [Luc06], with a number of results in the interim (see the references in the latter). The best bounds available

for general two-point configurations are summarised in a recent preprint of Lyall and Rice [LR15].

For non-linear configurations of length greater than two, the only existing quantitative result is due to Green [Gre02], who considers three-term progressions with difference equal to a sum of two squares. The logarithmic density of such numbers, together with their multiplicative structure, allows for methods unavailable for the sparser configurations considered in this paper. Employing Corollary 1.2 we obtain an alternative proof of Green's result.

The structure of our argument is discussed in detail in §2. In brief, our approach is to apply the method of van der Corput differencing to relate the non-linear configuration (1.1) to a longer linear configuration

$$x, \quad x + a_1 y, \quad \dots, \quad x + a_d y \tag{1.6}$$

with length $d = d(n, k)$ dependent only on n and k . We then treat this linear configuration using the methods of Gowers [Gow01]. The use of van der Corput's inequality allows us to control the size of the coefficients a_i . In essence, these deliberations establish that the polynomial progressions under consideration are controlled by an average of local Gowers norms, each localised to a subinterval.

The main technical difficulty is that the common difference y in the linear configuration (1.6) is constrained to lie in a much shorter interval than the shift parameter x . Unfortunately, the current inverse theory for the Gowers norms can only handle parameters x and y ranging over similarly sized intervals. Our strategy, heuristically at least, is to decompose y into a difference of smaller parameters $y = y_2 - y_1$. Changing variables in the shift x , we transform the configuration (1.6) into one of the form

$$x + b_1 y_1, \quad x + c_2 y_2, \quad x + b_3 y_1 + c_3 y_2, \quad \dots, \quad x + b_d y_1 + c_d y_2.$$

For each fixed value of x , one can view this as a shift of the linear configuration

$$b_1 y_1, \quad c_2 y_2, \quad b_3 y_1 + c_3 y_2, \quad \dots, \quad b_d y_1 + c_d y_2.$$

Crucially, in this linear configuration the parameters y_1 and y_2 range over the same interval. To each of these shifted 'short' configurations we apply Gowers's inverse theorem for the U^d -norm [Gow01], which yields a density increment on an even shorter subprogression.

We end this introduction by showing how Corollary 1.2 follows from Theorem 1.1.

Proof that Theorem 1.1 \implies Corollary 1.2. Suppose that $A \subset [N]$ lacks configurations of the form (1.4). Since K is a union of proper subspaces of \mathbb{R}^m , there exists $\mathbf{z} \in \mathbb{Z}^m \setminus K$. Notice that we must have $y\mathbf{z} \notin K$ for all $y \in \mathbb{Z} \setminus \{0\}$. Let us define $c_i := P_i(\mathbf{z})$ for $i = 1, \dots, n$. Then by homogeneity, the set A lacks configurations of the form $x, x + c_1 y^k, \dots, x + c_n y^k$ with $y \in \mathbb{Z} \setminus \{0\}$. The result now follows on employing Theorem 1.1. \square

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2. THE STRUCTURE OF OUR ARGUMENT

The structure of our argument closely follows the general density increment strategy of [Rot53, Gow01, Gre02]. Let us illustrate these ideas with respect to the configuration

$$x, \quad x + y^2, \quad x + 2y^2 \quad (y \in \mathbb{Z} \setminus \{0\}). \quad (2.1)$$

Our ultimate aim is to show that if a set $A \subset [N]$ of density $\delta := |A|/N$ lacks (2.1), then there exists a long arithmetic progression with square common difference

$$a + q^2 \cdot [N_1] \quad (2.2)$$

on which A has increased density. Let A_1 denote the set of $x \in [N_1]$ for which $a + q^2 x \in A$. Then the fact that (2.2) has square common difference ensures that A_1 also lacks (2.1), moreover A_1 has greater density on $[N_1]$ than A does on $[N]$. Iterating this argument eventually results in a configuration-free set whose density exceeds one. This contradiction allows us to extract a quantitative bound on the density of the initial set A . The proof of the density increment step occupies the majority of our paper, the more standard iteration and extraction of a final bound taking place in §7.

Given functions $f_i : \mathbb{Z} \rightarrow \mathbb{R}$ define the trilinear operator

$$T(f_0, f_1, f_2) := \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + y^2) f_2(x + 2y^2).$$

Then $T(1_A) := T(1_A, 1_A, 1_A)$ counts the number of configurations (2.1) in the set A , and we begin by comparing this to $T(\delta 1_{[N]})$, the expected value were A a random set of density δ . A crude lower bound gives

$$T(\delta 1_{[N]}) \gg \delta^3 N^{3/2}.$$

Hence if

$$|T(1_A) - T(\delta 1_{[N]})| \leq \frac{1}{2} |T(\delta 1_{[N]})| \quad (2.3)$$

then $T(1_A) \gg \delta^3 N^{3/2}$. In particular, $T(1_A) > 0$, which yields a contradiction if we are assuming that A lacks (2.1).

It follows that (2.3) does not hold, which amounts to assuming that

$$|T(1_A) - T(\delta 1_{[N]})| \gg \delta^3 N^{3/2}.$$

Write $f_A = 1_A - \delta 1_{[N]}$ for the balanced function of A . Then by trilinearity there must exist 1-bounded functions $f_i : \mathbb{Z} \rightarrow [-1, 1]$ supported on $[N]$, at least one of which is equal to f_A , and such that

$$|T(f_0, f_1, f_2)| \gg \delta^3 N^{3/2}. \quad (2.4)$$

For the sake of exposition, let us assume that $f_2 = f_A$. So far these deductions are standard, and closely follow [Gow01].

Definition (Gowers uniformity norm). Given a function $f : \mathbb{Z} \rightarrow \mathbb{R}$ with finite support, define

$$\|f\|_{U^d}^{2^d} := \sum_{h_1, \dots, h_d} \sum_x \Delta_{h_1, \dots, h_d} f(x), \quad (2.5)$$

where

$$\Delta_h f(x) := f(x+h)f(x) \quad (2.6)$$

and

$$\Delta_{h_1, \dots, h_d} f := \Delta_{h_1} \dots \Delta_{h_d} f.$$

For $S \subset \mathbb{Z}$ let us define the U^d -norm localised to S by

$$\|f\|_{U^d(S)} := \|f1_S\|_{U^d}.$$

Were one able to continue as in [Gow01], one might hope to prove that there exist absolute constants d and $c > 0$ such that

$$\sup_{|f_0|, |f_1| \leq 1_{[N]}} |T(f_0, f_1, f_2)| \ll T(1_{[N]}) \left(\frac{\|f_2\|_{U^d}}{\|1_{[N]}\|_{U^d}} \right)^c. \quad (2.7)$$

Green and Tao [GT08] call such a result a *generalised von Neumann theorem*. Combining this with (2.4) gives

$$\|f_A\|_{U^d} \gg \delta^C \|1_{[N]}\|_{U^d}. \quad (2.8)$$

Such a conclusion does not appear that useful unless $d = 1$. Unlike the relatively simple U^1 -(semi)norm

$$\|f\|_{U^1} = \left| \sum_x f(x) \right|,$$

the higher order U^d -norms are much harder to understand. However, the beef of [Gow01] says that largeness of these norms is explained, at least on a local level, by largeness of the U^1 -norm. More precisely, we have the following.

Gowers's inverse theorem. *For $d \geq 1$ there exist constants $C = C(d)$ and $c = c(d) > 0$ such that the following is true. Suppose that $f : \mathbb{Z} \rightarrow [-1, 1]$ satisfies*

$$\|f\|_{U^d[N]} \geq \delta \|1\|_{U^d[N]}.$$

Then one can partition $[N]$ into arithmetic progressions P_i , each of length at least $c\delta^C N^{c\delta^C}$, such that

$$\sum_i \|f\|_{U^1(P_i)} \geq c\delta^C \sum_i \|1\|_{U^1(P_i)}. \quad (2.9)$$

Employing this in conjunction with (2.8) provides a partition of $[N]$ into progressions P_i such that

$$\sum_i \left| \sum_{x \in P_i} f_A(x) \right| \gg \delta^C \sum_i |P_i|. \quad (2.10)$$

On noting that

$$\sum_i \sum_{x \in P_i} f_A(x) = \sum_x f_A(x) = 0,$$

we may add this to (2.10) to deduce that there exists an index i such that

$$\sum_{x \in P_i} f_A(x) \gg \delta^C |P_i|. \quad (2.11)$$

This yields a density increment on a subprogression.

There are two flaws with this argument: The first is that the subprogressions given by Gowers's inverse theorem may not have square common difference as in (2.2). Rectifying this requires a purely technical modification of [Gow01], as first demonstrated for the U^3 -norm by Green [Gre02]. This is explained further in §6.

The second flaw, and most problematic, is that no generalised von Neumann inequality of the form (2.7) exists in the literature. In recent work of Tao and Ziegler [TZ16], a qualitative version of such a result is deduced which amounts to saying that if $|T(f_0, f_1, f_A)|$ is large, then some global Gowers norm $\|f_A\|_{U^d}$ must also be large¹. However, the quantitative dependence in this is at least tower-exponential [TZ15], and is thus insufficient for our purpose.

The key idea of this paper is to aim for less. Instead of showing that the counting operator T is controlled by a single global Gowers norm, we show that T is controlled by an average of local Gowers norms, each localised to a subprogression of length approximately \sqrt{N} .

Definition (Localised U^d -norm). Define the U^d -norm localised to scale M by

$$\|f\|_{U^d \sim M} := \sum_x \|f\|_{U^d(x+[M])}. \quad (2.12)$$

This is an average of the Gowers norm of f over every interval of length M . A more complicated version of this localised norm appears in work of Tao and Ziegler [TZ08], and one can think of (2.12) as a version of their norm in which a number of extra averaging parameters have been fixed.

Using this norm we are able to prove the following local von Neumann theorem.

¹See also the video lecture: T. Tao, *Concatenation theorems for the Gowers uniformity norms*, BIRS workshop on Combinatorics Meets Ergodic Theory, <http://goo.gl/UskoBQ>.

Local von Neumann theorem. *Let f_0, f_1 be 1-bounded functions supported on $[N]$. Suppose that*

$$|T(f_0, f_1, f_A)| \geq \delta T(1_{[N]}).$$

Then, provided that $N \geq C\delta^{-C}$, there exists M in the range

$$\delta^C \sqrt{N} \ll M \ll \delta^{-C} \sqrt{N}$$

such that we have the local non-uniformity estimate

$$\|f_A\|_{U^7 \sim M} \gg \delta^C \|1_{[N]}\|_{U^7 \sim M}. \quad (2.13)$$

The non-uniformity estimate (2.13) can be interpreted as saying that, for at least $c\delta^C N$ of the intervals $x + [M]$, we have

$$\|f_A\|_{U^7(x+[M])} \gg \delta^C \|1\|_{U^7(x+[M])}. \quad (2.14)$$

To each of these intervals, we apply Gowers's inverse theorem (suitably modified) to deduce the existence of a partition of $x + [M]$ into fairly long progressions $P_{x,i}$, each with square common difference, and such that

$$\sum_i \|f_A\|_{U^1(P_{x,i})} \gg \delta^C \sum_i \|1_{[N]}\|_{U^1(P_{x,i})}.$$

Taking the trivial partition for the remaining intervals, it follows that for all x there exists a partition of $x + [M]$ into progressions $P_{x,i}$ with square common difference such that

$$\sum_x \sum_i \left| \sum_{y \in P_{x,i}} f_A(y) \right| \gg \delta^C \sum_x \sum_i |P_{x,i}|. \quad (2.15)$$

Crucially, since f_A has mean zero, we have

$$\begin{aligned} \sum_x \sum_i \sum_{y \in P_{x,i}} f_A(y) &= \sum_{z \in [M]} \sum_x f_A(x+z) \\ &= 0. \end{aligned}$$

We may therefore add this quantity to (2.15) to conclude that there exists a long arithmetic progression $P_{x,i}$ with square common difference such that

$$\sum_{y \in P_{x,i}} f_A(y) \gg \delta^C |P_{x,i}|.$$

This yields the required density increment.

In the remainder of this section, we outline the ideas behind the local von Neumann theorem. The inspiration for our approach is an argument of Green–Tao–Ziegler [Tao13] which establishes a local von Neumann theorem for the two-point configuration $x, x + y^2$, showing that it is controlled by an average of local U^1 -norms. We begin by sketching their argument.

Let f_0 be a 1-bounded function supported on $[N]$. Then we are interested in bounding the quantity

$$\sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_A(x + y^2).$$

Write I for the interval $[\sqrt{N}]$. By an application of the Cauchy–Schwarz inequality and a change of variables we have

$$\begin{aligned} \left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_A(x + y^2) \right|^2 &\leq N \sum_x \sum_{y_1, y_2 \in I} f_A(x + y_1^2) f_A(x + y_2^2) \\ &= N \sum_{|h| < \sqrt{N}} \sum_x f_A(x - h^2) \sum_{y \in I \cap (I - h)} f_A(x + 2hy) \\ &\ll N^{3/2} \max_{|h| < \sqrt{N}} \sum_x \|f_A\|_{U^1(x + P_h)}, \end{aligned}$$

where P_h is the progression $\{2hy : y \in I \cap (I - h)\}$. Here we have made use of the simple identity $(y + h)^2 - y^2 = 2hy + h^2$.

As stated, there are two deficiencies with this local von Neumann inequality, in that the common difference of the progression P_h may be zero or a non-square. Both of these difficulties can be surmounted by replacing the Cauchy–Schwarz inequality with van der Corput’s inequality (see §3 for a statement of this inequality). In doing so, one can deduce that for any $H \leq \sqrt{N}$ we have

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_A(x + y^2) \right|^2 \ll \frac{N^3}{H} + N^{3/2} \max_{0 < |h| < H} \|f_A\|_{U^1(x + P_h)}.$$

This ensures that the common difference of P_h is non-zero, yet it still may be a non-square. However, since we can control the size of this common difference (it is bounded above by $2H$), we can partition P_h into at most $2H$ further subprogressions of square common difference. It follows that there exists a progression P with square common difference such that

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_A(x + y^2) \right|^2 \ll \frac{N^3}{H} + N^{3/2} H \|f_A\|_{U^1(x + P)}. \quad (2.16)$$

Taking $H = C\delta^{-C}$ in (2.16), the assumption

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_A(x + y^2) \right| \geq \delta N^{3/2}$$

implies that

$$\sum_x \|f_A\|_{U^1(x + P)} \gg \delta^C N^{3/2}. \quad (2.17)$$

Employing the trivial estimate $|f_A| \leq 1_{[N]}$ and assuming that $N \geq C\delta^{-C}$, the left-hand side of (2.17) is at most $O(N|P|)$, which gives the lower bound

$$|P| \gg \delta^C N^{1/2}.$$

The corresponding upper bound $|P| \leq \sqrt{N}$ follows since $P \subset \{2hy : y \in [\sqrt{N}]\}$. We have therefore deduced the following.

Two-point local von Neumann. *Let f_0 be a 1-bounded function supported on $[N]$. Suppose that*

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_A(x + y^2) \right| \geq \delta N^{3/2}.$$

Then, provided that $N \geq C\delta^{-C}$, there exists a progression P with square common difference and length

$$\delta^C \sqrt{N} \ll |P| \leq \sqrt{N}$$

such that we have the local non-uniformity estimate

$$\sum_x \|f_A\|_{U^1(x+P)} \gg \delta^C \sum_x \|1_{[N]}\|_{U^1(x+P)}.$$

For longer configurations, one must employ the van der Corput inequality considerably more times. Let us illustrate this for the inhomogeneous counting operator

$$\tilde{T}(f_0, f_1, f_A) := \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + y) f_2(x + y^2). \quad (2.18)$$

A single application of van der Corput's inequality gives some $0 < |h_1| < H$ and some interval $I \subset [\sqrt{N}]$ for which

$$|\tilde{T}(f_0, f_1, f_A)|^2 \ll \frac{N^3}{H} + N^{3/2} \left| \sum_x \sum_{y \in I} f_1(x) f_1(x + h_1) f_2(x + y^2 - y) f_2(x + (y + h_1)^2 - y) \right|.$$

To save on notation, let us write \tilde{f} for a function of the form $x \mapsto f(x+b)$ for some fixed integer b . Different occurrences of \tilde{f} in the same equation may refer to different values of b , but no confusion should arise. A second application of van der Corput's inequality gives the existence of some $0 < |h_2| < H$ and a second interval $I' \subset I$ such that

$$|\tilde{T}(f_0, f_1, f_2)|^4 \ll \frac{N^6}{H} + N^{9/2} \left| \sum_x \sum_{y \in I'} f_2(x) \tilde{f}_2(x + 2h_1 y) \tilde{f}_2(x + 2h_2 y) \tilde{f}_2(x + 2(h_1 + h_2)y) \right|.$$

Suppose that

$$|\tilde{T}(f_0, f_1, f_A)| \geq \delta N^{3/2}. \quad (2.19)$$

Then as before, taking $H := C\delta^{-C}$, we can conclude the existence of non-zero integers $|a_i| \ll \delta^{-C}$ and $M \leq \sqrt{N}$ such that

$$\left| \sum_x \sum_{y \in [M]} f_A(x) \tilde{f}_A(x + a_1 y) \tilde{f}_A(x + a_2 y) \tilde{f}_A(x + a_3 y) \right| \gg \delta^C N^{3/2}. \quad (2.20)$$

The trivial estimate $|f_A| \leq 1_{[N]}$ also yields the lower bound $M \gg \delta^C \sqrt{N}$. Our next step is to convert (2.20) into a local non-uniformity estimate. Let us demonstrate how this is done for the simpler linear average

$$\left| \sum_x \sum_{y \in [M]} f_A(x) f_A(x + y) f_A(x - y) \right| \gg \delta^C N M. \quad (2.21)$$

Let $M_1 \leq M$. Then one can re-write the inner sum in (2.21) as within $O(M_1)$ of

$$\begin{aligned} M_1^{-2} \sum_{y_1, y_2 \in [M_1]} \sum_{y \in [M] - y_1 + y_2} f_A(x) f_A(x + y) f_A(x - y) = \\ M_1^{-2} \sum_{y_1, y_2 \in [M_1]} \sum_{y \in [M]} f_A(x) f_A(x + y - y_1 + y_2) f_A(x - y + y_1 - y_2). \end{aligned}$$

Changing variables in x and maximising over y , we deduce that the left-hand side of (2.21) is at most

$$\frac{M}{M_1^2} \left| \sum_x \sum_{y_1, y_2 \in [M_1]} f_A(x + y_1) \tilde{f}_A(x + y_2) \tilde{f}_A(x + 2y_1 - y_2) \right| + O(NM_1).$$

Taking $M_1 = c\delta^C M$, (2.21) implies that

$$\left| \sum_x \sum_{y_1, y_2 \in [M_1]} f_A(x + y_1) \tilde{f}_A(x + y_2) \tilde{f}_A(x + 2y_1 - y_2) \right| \gg \delta^C N M_1^2 \quad (2.22)$$

For fixed x define the functions $g_1(y) := f_A(x + y)1_{[M_1]}(y)$, $g_2(y) := \tilde{f}_A(x + y)1_{[M_1]}(y)$ and $h(y) := \tilde{f}_A(x + y)1_{[-2M_1, 2M_1]}(y)$. Then by orthogonality

$$\sum_{y_1, y_2 \in [M_1]} g_1(y_1) g_2(y_2) h(2y_1 - y_2) = \int_{\mathbb{T}} \hat{g}_1(-2\alpha) \hat{g}_2(\alpha) \hat{h}(\alpha) d\alpha,$$

where we have defined the Fourier transform by

$$\hat{g}(\alpha) := \sum_x g(x) e(\alpha x). \quad (2.23)$$

Using Hölder's inequality, Parseval and the (easily checked) identity $\|g\|_{U^2(\mathbb{Z})} = \|\hat{g}\|_{L^4(\mathbb{T})}$, we deduce that

$$\sum_{y_1, y_2 \in [M_1]} g_1(y_1)g_2(y_2)h(2y_1 - y_2) \leq \|g_1\|_{L^2(\mathbb{Z})} \|g_2\|_{U^2(\mathbb{Z})} \|h\|_{U^2(\mathbb{Z})}.$$

Since $\|g_1\|_{L^2(\mathbb{Z})} \leq M_1^{1/2}$ and $\|g_2\|_{U^2(\mathbb{Z})} \leq M_1^{3/4}$ and $\delta^C \sqrt{N} \ll M_1 \leq \sqrt{N}$ we can set $I = [-2M_1, 2M_1]$ and conclude that

$$\sum_x \|f_A\|_{U^2(x+I)} \gg \delta^C N^{11/4}$$

for some interval I satisfying $\delta^C \sqrt{N} \ll |I| \ll \delta^{-C} \sqrt{N}$. This is the local von Neumann estimate we hope to prove.

A similar argument can be made to work for the longer linear average (2.20), replacing the U^2 -norm with the U^3 -norm. The general argument for arbitrarily long linear configurations is carried out in §5.

This linearisation process, which takes a large non-linear polynomial average such as (2.19), and converts it into a large linear average (2.20), is generalisable and carried out in detail in §3. As the complexity of the polynomial configuration increases, the number of applications of van der Corput's inequality required increases inordinately. Ensuring that this process does in fact terminate requires a fairly abstract inductive scheme to be carried out, a scheme which is essentially the PET-induction of Bergelson–Leibman [BL96]. The number of steps in this process grows so rapidly that, for general configurations, we have refrained from estimating the explicit quantitative dependence in the absolute constant C appearing in the deduction of (2.20) from (2.19). This ultimately leads to the absence of a lower bound for $c(n, k)$ in Theorem 1.1, in contrast to Gowers's estimate (1.5).

Although we have sketched how to prove a local von Neumann theorem for the inhomogeneous configuration

$$x, \quad x + y, \quad x + y^2, \tag{2.24}$$

this configuration is not covered by Theorem 1.1. More generally, we demonstrate in §§3–5 that a local von Neumann theorem can be proved for *any* configuration of the form $x, x + P_1(y), \dots, x + P_n(y)$, where $P_i \in \mathbb{Z}[y]$. The main obstacle to obtaining density bounds for sets lacking inhomogeneous configurations such as (2.24) is the density increment step. To see this, note that if $A \subset [N]$ lacks (2.24) and has a density increment on progression of the form $x + q \cdot [N_1]$, then defining A_1 to be the set of $y \in [N_1]$ such that $x + qy \in A$, we see that A_1 lacks configurations of the form

$$x, \quad x + y, \quad x + qy^2 \quad (y \in \mathbb{Z} \setminus \{0\}). \tag{2.25}$$

The problem here is that the methods we have discussed (primarily Gowers's inverse theorem) deliver an increment on a progression with common difference q which is likely to be *much* larger than its length N_1 . As a

result, *every* subset of $[N_1]$ lacks the configuration (2.25), and there is no possibility of iterating the density increment argument.

The remaining sections are occupied with proving a rigorous version of the sketch outlined above.

3. VAN DER CORPUT DIFFERENCING

The aim of this section and its sequel is to show how a large non-linear average

$$\sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \quad (3.1)$$

leads to a large *linear* average, albeit over a longer configuration. This deduction proceeds via van der Corput differencing, each application of which bounds a polynomial average such as (3.1) by a polynomial average of smaller degree. The precise notion of degree is introduced in §4; in this section we confine ourselves to describing the differencing step.

Lemma 3.1 (van der Corput inequality). *Let $g : \mathbb{Z} \rightarrow \mathbb{C}$ be a function supported on a finite set $\mathcal{S} \subset \mathbb{Z}$. Given a finite set $\mathcal{H} \subset \mathbb{Z}$, write $r_{\mathcal{H}}(h)$ for the number of pairs $(h_1, h_2) \in \mathcal{H}^2$ such that $h_1 - h_2 = h$. Then we have the estimate*

$$\left| \sum_y g(y) \right|^2 \leq \frac{|\mathcal{S} - \mathcal{H}|}{|\mathcal{H}|^2} \sum_h r_{\mathcal{H}}(h) \sum_y g(y+h) \overline{g(y)}.$$

Proof. By a change of variables, for any $h \in \mathbb{Z}$ we have

$$\sum_y g(y) = \sum_y g(y+h).$$

Averaging over $h \in \mathcal{H}$ and interchanging the order of summation gives

$$\sum_y g(y) = \frac{1}{|\mathcal{H}|} \sum_y \sum_{h \in \mathcal{H}} g(y+h).$$

The function

$$y \mapsto \sum_{h \in \mathcal{H}} g(y+h)$$

is supported on the difference set $\mathcal{S} - \mathcal{H}$. Squaring and applying Cauchy–Schwarz, we deduce that

$$\begin{aligned} \left| \sum_y g(y) \right|^2 &\leq \frac{|\mathcal{S} - \mathcal{H}|}{|\mathcal{H}|^2} \sum_y \sum_{h_1, h_2 \in \mathcal{H}} g(y+h_1) \overline{g(y+h_2)} \\ &= \frac{|\mathcal{S} - \mathcal{H}|}{|\mathcal{H}|^2} \sum_{h_1, h_2 \in \mathcal{H}} \sum_y g(y+h_1-h_2) \overline{g(y)} \\ &= \frac{|\mathcal{S} - \mathcal{H}|}{|\mathcal{H}|^2} \sum_h r_{\mathcal{H}}(h) \sum_y g(y+h) \overline{g(y)}. \end{aligned}$$

□

Lemma 3.2 (weak van der Corput). *Suppose that $g : \mathbb{Z}^2 \rightarrow [-1, 1]$ is supported on $[N] \times [M]$ with $N, M \geq 1$. Let $1 \leq H \leq M$ and let $\mathcal{H}_1 \subset \mathbb{Z}$ denote a set containing 0. Then there exists $h \in [H] \setminus \mathcal{H}_1$ such that*

$$\sum_x \left(\sum_y g(x, y) \right)^2 \ll \frac{NM^2 |\mathcal{H}_1|}{H} + M \sum_{x, y} g(x, y + h) g(x, y).$$

Proof. Let us apply Lemma 3.1 with $g_x(y) := g(x, y)$, $\mathcal{S} := [M]$ and $\mathcal{H} := [H]$, giving

$$\sum_x \left(\sum_y g(x, y) \right)^2 \leq \frac{2M}{[H]} \sum_h \frac{r_{[H]}(h)}{[H]} \sum_{x, y} g(x, y + h) g(x, y).$$

A change of variables yields the identity

$$\sum_{x, y} g(x, y + h) g(x, y) = \sum_{x, y} g(x, y) g(x, y - h).$$

Combining this with the fact that $r_{[H]}(0) = [H]$, $r_{[H]}(-h) = r_{[H]}(h)$ and $r_{[H]}(h) = 0$ if $|h| \geq H$, we have

$$\begin{aligned} \sum_h \frac{r_{[H]}(h)}{[H]} \sum_{x, y} g(x, y + h) g(x, y) &= \\ \sum_{x, y} g(x, y)^2 + \sum_{h \in [H]} \frac{2r_{[H]}(h)}{[H]} \sum_{x, y} g(x, y + h) g(x, y). \end{aligned} \quad (3.2)$$

Using the trivial estimates $|\text{supp}(g)| \leq NM$, $r_{[H]}(h) \leq [H]$ and $|\mathcal{H}_1 \cap [H]| \leq |\mathcal{H}_1| - 1$, the right-hand side of (3.2) is at most

$$NM + 2(|\mathcal{H}_1| - 1)NM + \sum_{h \in [H] \setminus \mathcal{H}_1} \frac{2r_{[H]}(h)}{[H]} \sum_{x, y} g(x, y + h) g(x, y)$$

By the pigeon-hole principle and monotonicity of expectation, there exists $h' \in [H] \setminus \mathcal{H}_1$ such that

$$\sum_{h \in [H] \setminus \mathcal{H}_1} \frac{r_{[H]}(h)}{[H]} \sum_{x, y} g(x, y + h) g(x, y) \leq [H] \sum_{x, y} g(x, y + h') g(x, y).$$

The required inequality follows. □

One can think of the set \mathcal{H}_1 as those ‘bad’ differencing parameters h we wish to avoid.

Lemma 3.3 (Linearisation step). *Let $f_0, f_1, \dots, f_n : \mathbb{Z} \rightarrow [-1, 1]$ be 1-bounded functions supported on $[N]$, let I be an interval of at most M*

integers, let \mathcal{H}_1 be a set containing 0 and let $P_1, \dots, P_n : \mathbb{Z} \rightarrow \mathbb{Z}$. Then for any $H \leq M$ there exists $h \in [H] \setminus \mathcal{H}_1$ such that

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right| \ll \left(\frac{|\mathcal{H}_1|}{H} \right)^{1/2} + \left(\frac{1}{NM} \sum_x \sum_{y \in I \cap (I-h)} \prod_{\substack{1 \leq i \leq n \\ \omega \in \{0,1\}}} f_i(x + P_i(y + \omega h) - P_1(y)) \right)^{1/2}.$$

Proof. Shifting the argument of the functions P_i if necessary, we may assume that $I \subset [M]$. Set $g(x, y) := f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) 1_{[N]}(x) 1_I(y)$ and let $H \in [1, M]$ (to be determined later). Then by the Cauchy–Schwarz inequality and Lemma 3.2 there exists $h \in [H] \setminus \mathcal{H}_1$ such that

$$\begin{aligned} \left(\sum_x f_0(x) \sum_y g(x, y) \right)^2 &\leq \left(\sum_x f_0(x)^2 \right) \sum_x \left(\sum_y g(x, y) \right)^2 \\ &\ll N^2 M^2 |\mathcal{H}_1| H^{-1} + NM \sum_{x, y} g(x, y + h) g(x, y). \end{aligned}$$

□

4. THE LINEARISATION PROCESS

In this section we iteratively apply Lemma 3.3, beginning with the configuration $x, x + c_1 y^k, \dots, x + c_n y^k$ and eventually obtaining a configuration of the form $x, x + a_1 y, \dots, x + a_d y$. The complexity of the intermediate configurations requires us to take an abstract approach. Moreover, each application of the linearisation step necessitates a number of technical assumptions whose sole purpose is to guarantee that the coefficients a_i in our final linear configuration are non-zero and distinct. Before proceeding to describe the argument in general, we illustrate the underlying ideas for the configuration $x, x + y^2, x + 2y^2$.

Lemma 4.1 (Linearisation for square 3APs). *Let $f_0, f_1, f_2 : \mathbb{Z} \rightarrow [-1, 1]$ be supported on $[N]$ and let $1 \leq H \leq \sqrt{N}$. Then there exists an interval $I \subset [\sqrt{N}]$ and integers a_i, b_i with the a_i distinct, $1 \leq a_i \ll H$ and such that*

$$\begin{aligned} \left| N^{-3/2} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + y^2) f_2(x + 2y^2) \right| &\ll \\ H^{-1/8} + \left| N^{-3/2} \sum_x \sum_{y \in I} f_2(x) f_2(x + a_1 y + b_1) \cdots f_2(x + a_7 y + b_7) \right|^{1/8}. \end{aligned} \tag{4.1}$$

Notation. To avoid lengthy expressions, we write \tilde{f} for a function of the form $x \mapsto f(x + b)$ for some integer b . Different occurrences of \tilde{f} in the

same equation may refer to different values of b , but no confusion should arise.

Proof. Our assumption on the support of f_i ensures that $f_0(x)f_1(x+y^2)f_2(x+2y^2) \neq 0$ only when $y \in [\sqrt{N}]$. Write I for this interval, and M for the number of integers it contains. By Lemma 3.3 with $\mathcal{H}_1 = \{0\}$ there exists an integer $1 \leq h_1 \leq H$ and an interval $I_1 \subset I$ satisfying

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I} f_0(x) f_1(x+y^2) f_2(x+2y^2) \right| \ll \frac{1}{H^{1/2}} + \left(\frac{1}{NM} \sum_x \sum_{y \in I_1} f_1(x) \tilde{f}_1(x+2h_1y) \tilde{f}_2(x+y^2) \tilde{f}_2(x+(y+h_1)^2+2h_1y) \right)^{1/2}. \quad (4.2)$$

Re-applying Lemma 3.3 with $\mathcal{H}_2 := \{0\}$, we may conclude that there exists an integer $1 \leq h_2 \leq H$ and an interval $I_2 \subset I_1$ satisfying

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I} f_0(x) f_1(x+y^2) f_2(x+2y^2) \right| \ll \frac{1}{H^{1/2}} + \frac{1}{H^{1/4}} + \left(\frac{1}{NM} \sum_x \sum_{y \in I_2} F_1(x, y) \right)^{1/4}.$$

where $F_1(x, y)$ is equal to

$$f_1(x) \tilde{f}_1(x) \tilde{f}_2(x+y^2-2h_1y) \tilde{f}_2(x+(y+h_2)^2-2h_1y) \times \tilde{f}_2(x+(y+h_1)^2) \tilde{f}_2(x+(y+h_1+h_2)^2).$$

A function of the form $x \mapsto f_1(x) \tilde{f}_1(x)$ is 1-bounded, supported on $[N]$ and independent of y . We may therefore remove this function by re-applying Lemma 3.3 with $\mathcal{H}_3 := \{0\}$ to conclude that there exists an integer $1 \leq h_3 \leq H$ and an interval $I_3 \subset I_2$ satisfying

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I} f_0(x) f_1(x+y^2) f_2(x+2y^2) \right| \ll \frac{1}{H^{1/2}} + \frac{1}{H^{1/4}} + \frac{1}{H^{1/8}} + \left(\frac{1}{NM} \sum_x \sum_{y \in I_3} F_2(x, y) \right)^{1/8}. \quad (4.3)$$

where $F_2(x, y)$ is equal to

$$f_2(x) \tilde{f}_2(x+2h_3y) \tilde{f}_2(x+2h_2y) \tilde{f}_2(x+2(h_2+h_3)y) \tilde{f}_2(x+4h_1y) \times \tilde{f}_2(x+2(2h_1+h_3)y) \tilde{f}_2(x+2(h_1+h_2)y) \tilde{f}_2(x+2(2h_1+h_2+h_3)y).$$

The lemma is complete, provided the following coefficients are all distinct

$$2h_1, \quad h_2, \quad h_3, \quad 2h_1 + h_2, \quad 2h_1 + h_3, \quad h_2 + h_3, \quad 2h_1 + h_2 + h_3.$$

Unfortunately, we cannot guarantee this with the proof as written. However, distinctness would follow if instead of taking $\mathcal{H}_2 = \mathcal{H}_3 = \{0\}$ we took

$$\mathcal{H}_2 := \{0, 2h_1\}, \quad \mathcal{H}_3 := \{0, 2h_1, h_2, 2h_1 + h_2, h_2 - 2h_1, 2h_1 - h_2\}.$$

Lemma 3.3 permits this, increasing the absolute constant in (4.3) by a factor of at most $6^{1/8}$. \square

The above argument required three applications of Lemma 3.3 in order to linearise the simplest example of a non-linear k th power configuration of length greater than two. In general we require many more applications of the linearisation step, and at each stage of the iteration, it is not immediately obvious that we have reduced the ‘degree’ of the configuration at all. To see that we have indeed reduced an invariant associated to the configuration, we require the following definition.

Definition (Degree sequence). Given polynomials $P_1, \dots, P_n \in \mathbb{Z}[x]$, let $L(P_i)$ denote the leading coefficient of P_i and define

$$D_r(P_1, \dots, P_n) := \# \{L(P_i) : \deg P_i = r\}.$$

In words, $D_r(\mathbf{P})$ is the number of distinct leading coefficients occurring amongst the degree r polynomials in \mathbf{P} . Let us define the *degree sequence* of $\mathbf{P} = (P_1, \dots, P_n)$ by

$$D(\mathbf{P}) := (D_1(\mathbf{P}), D_2(\mathbf{P}), D_3(\mathbf{P}), \dots).$$

Definition (Colex order). We order degree sequences according to the colexicographical ordering, so that $D(\mathbf{P}) \prec D(\mathbf{Q})$ if there exists $r \in \mathbb{N}$ such that $D_r(\mathbf{P}) < D_r(\mathbf{Q})$ and for all $s > r$ we have $D_s(\mathbf{P}) = D_s(\mathbf{Q})$.

Lemma 4.2. *Let \mathcal{S} denote the set of sequences $(m_i)_{i \in \mathbb{N}}$ of non-negative integers with all but finitely many entries equal to zero. Then colex induces a well-ordering on \mathcal{S} . In particular, if $P(\mathbf{m})$ is a proposition defined on \mathcal{S} satisfying*

$$[(\forall \mathbf{m}' \prec \mathbf{m}) P(\mathbf{m}')] \implies P(\mathbf{m}),$$

then $P(\mathbf{m})$ is true for all $\mathbf{m} \in \mathcal{S}$.

Proof. We leave the reader to check that \preceq is transitive, anti-symmetric and total. We show that every non-empty subset of \mathcal{S} has a least element.

Let \mathcal{F} be a non-empty subset of \mathcal{S} . We construct a sequence $(m_l^*)_{l \in \mathbb{N}}$ such that for each $k \in \mathbb{N}$ the set

$$\mathcal{F}_k := \{\mathbf{m} \in \mathcal{F} : m_l = m_l^* \text{ for all } l \geq k\}$$

is non-empty and for any $\mathbf{m} \in \mathcal{F}$ we have

$$(m_l^*)_{l \geq k} \preceq (m_l)_{l \geq k}. \tag{4.4}$$

It follows that \mathbf{m}^* is a least element of \mathcal{F} .

Since \mathcal{F} is non-empty, there exists $\mathbf{m} \in \mathcal{F}$. Write k_0 for the minimum index satisfying $m_l = 0$ for all $l \geq k_0$. Then we take $m_l^* := 0$ for all $l \geq k_0$.

Suppose we have constructed $(m_l^*)_{l \geq k}$ with the required properties and $k > 1$. Writing π_{k-1} for the projection onto the $(k-1)$ coordinate, $\pi_{k-1}(\mathcal{F}_k)$ is a non-empty set of non-negative integers, hence contains a least element m_{k-1}^* .

Letting $\mathbf{m} \in \mathcal{F}$, we wish to check that

$$(m_l^*)_{l \geq k-1} \preceq (m_l)_{l \geq k-1}. \quad (4.5)$$

Since (4.4) holds, we are done if the inequality in (4.4) is strict. We may therefore assume that

$$(m_l^*)_{l \geq k} = (m_l)_{l \geq k}.$$

The inequality (4.5) now follows since $m_{k-1} \in \pi_{k-1}(\mathcal{F}_k)$, and therefore $m_{k-1}^* \leq m_{k-1}$. \square

For the next two lemmas we assume that $f_0, f_1, \dots, f_n : \mathbb{Z} \rightarrow [-1, 1]$ are 1-bounded functions supported on $[N]$, that I is an interval of at most M integers, and that $P_1, \dots, P_n \in \mathbb{Z}[x]$ are polynomials of height at most H , maximal degree k and such that $0, P_1, \dots, P_n$ have distinct non-constant parts.

Lemma 4.3 (Degree sequence inductive step). *Suppose that $k > 1$. Then for any $H_1 \leq M$ there exists an interval $I' \subset I$ along with 1-bounded functions $g_0, \dots, g_{n'}$ supported on $[N]$ and polynomials $Q_1, \dots, Q_{n'}$ with $D(\mathbf{Q}) \prec D(\mathbf{P})$ such that*

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right| \ll n H_1^{-1/2} + \left| \frac{1}{NM} \sum_x \sum_{y \in I'} g_0(x) g_1(x + Q_1(y)) \cdots g_{n'}(x + Q_{n'}(y)) \right|^{1/2}. \quad (4.6)$$

Moreover, we can ensure that

- $g_{n'} = f_n$,
- $n' \leq 2n$,
- $0, Q_1, \dots, Q_{n'}$ have distinct non-constant parts,
- $Q_1, \dots, Q_{n'}$ have height at most $H(4H_1)^k$,
- writing t for the smallest degree such that $D_t(\mathbf{P}) > 0$, we have

$$D(\mathbf{Q}) = (i_1, \dots, i_{t-1}, D_t(\mathbf{P}) - 1, D_{t+1}(\mathbf{P}), D_{t+2}(\mathbf{P}), \dots) \quad (4.7)$$

for some $i_1 + \dots + i_{t-1} \leq 2n$.

Proof. At the cost of increasing the height H by a factor of 2, we may assume that the polynomial P_n occurring in the argument of the function f_n has maximal degree $k > 1$. To see why this is so, suppose that the maximal index j with $\deg P_j = k$ satisfies $j < n$. Then performing the

change of variables $x \mapsto x - P_j(y)$ results in a configuration $\tilde{\mathbf{P}}$ of the required form satisfying

$$\begin{aligned} \sum_x \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \\ = \sum_x \sum_{y \in I} f_j(x) f_1(x + \tilde{P}_1(y)) \cdots f_0(x + \tilde{P}_j(y)) \cdots f_n(x + \tilde{P}_n(y)) \end{aligned}$$

Moreover, $\tilde{\mathbf{P}}$ has height at most $2H$ and $0, \tilde{P}_1, \dots, \tilde{P}_n$ have distinct non-constant parts.

Given this assumption, let us re-arrange the remaining indices with respect to the order of $\deg P_i$, so that there exists an index $l \leq n$ with

$$\deg P_i = 1 \iff i < l \quad \text{and} \quad \deg P_l = \min \{ \deg P_i : i \geq l \}. \quad (4.8)$$

Claim. *There are at most n^2 choices of h for which the following polynomials have indistinct non-constant parts*

$$P_1(y), \dots, P_n(y), P_l(y + h), \dots, P_n(y + h) \quad (4.9)$$

To establish the claim, let us suppose that h is such that two of the polynomials in the list (4.9) have the same non-constant part. Since the polynomials P_1, \dots, P_n have distinct non-constant parts, the only possibility is that $P_i(y + h) - P_j(y)$ is constant for some $l \leq i \leq n$ and $1 \leq j \leq n$. Let $P_i(y) = a_d y^d + a_{d-1} y^{d-1} + \dots$ with $a_d \neq 0$. Then since $d > 1$ we have

$$P_i(y + h) = a_d y^d + (da_d h + a_{d-1}) y^{d-1} + \dots$$

The expression $da_d h + a_{d-1}$ must equal the coefficient of y^{d-1} in P_j , and this completely determines h . Since there are at most n choices for P_j and at most n choices for P_i the claim follows.

Let \mathcal{H}_1 denote the set of h for which two of the polynomials in (4.9) have the same non-constant part. Notice that \mathcal{H}_1 contains 0. Applying Lemma 3.3, we deduce that there exists $h \in [H_1] \setminus \mathcal{H}_1$ and an interval $I' \subset I$ such that the left-hand side of (4.6) is of order at most

$$nH_1^{-1/2} + \left| \frac{1}{NM} \sum_x \sum_{y \in I'} \prod_{\substack{1 \leq i \leq n \\ \omega \in \{0,1\}}} f_i(x + P_i(y + \omega h) - P_1(y)) \right|^{1/2}. \quad (4.10)$$

Using the notation (2.6) we see that for each $i < l$ there exists an integer $a_i = P_i(h)$ such that

$$f_i(x + P_i(y) - P_1(y)) f_i(x + P_i(y + h) - P_1(y)) = \Delta_{a_i} f_i(x + (P_i - P_1)(y)).$$

In this case, let us write $g_{i-1} := \Delta_{a_i} f_i$ and $Q_{i-1} := P_i - P_1$. For the remaining indices, we set

- $g_{l+2i-1} = g_{l+2i} := f_{l+i}$,
- $Q_{l+2i-1}(y) := P_{l+i}(y) - P_1(y)$,
- $Q_{l+2i}(y) := P_{l+i}(y + h) - P_1(y)$;

where in each case i ranges over $0 \leq i \leq n - l$. Then one can check that

$$g_0(x)g_1(x + Q_1(y)) \cdots g_{n'}(x + Q_{n'}(y)) = \prod_{\substack{1 \leq i \leq n \\ \omega \in \{0,1\}}} f_i(x + P_i(y + \omega h) - P_1(y)),$$

which yields (4.6) with $n' = 2n - l \leq 2n$.

From our claim we see that $0, Q_1, \dots, Q_{n'}$ have distinct non-constant parts, since adding P_1 to each polynomial in this sequence gives the sequence (4.9). Also $g_{n'} = g_{l+2(n-l)+1} = f_n = f_A$. From a crude estimate using the binomial theorem, one can check that the height of each Q_i is at most $\frac{1}{2}H(4H_1)^k$.

It remains to show that $D(\mathbf{Q})$ has the form given in (4.7), and consequently $D(\mathbf{Q}) \prec D(\mathbf{P})$. From (4.8) we have $t = \deg P_1$. Hence if $\deg P_i > t$ then for either choice of $\omega \in \{0, 1\}$, the polynomial $P_i(y + \omega h) - P_1(y)$ has the same leading term as P_i . It follows that for $s > t$ we have $D_s(\mathbf{Q}) = D_s(\mathbf{P})$. Let $\{a_1, \dots, a_r\}$ denote the set of leading coefficients which appear in some P_i with $\deg P_i = t$. We may assume that a_1 is the leading coefficient of P_1 . Then the set of leading coefficients occurring amongst those Q_i with $\deg Q_i = t$ is equal to $\{a_2 - a_1, \dots, a_r - a_1\}$, which has cardinality one less than $\{a_1, \dots, a_r\}$. Moreover, since there are at most $2n$ polynomials Q_i , the number of Q_i with $\deg Q_i < t$ is also at most $2n$. This leads to the bound for $i_1 + \dots + i_{t-1}$ claimed in the theorem. \square

Lemma 4.4 (Full linearisation). *Writing $\mathbf{m} := D(\mathbf{P})$, there exist positive integers $R = R(n, \mathbf{m})$, $r \leq R$ and $d \leq 2^r n$ such that for any $H_1 \leq M$ there exists an interval $I' \subset I$ along with 1-bounded functions g_0, \dots, g_d supported on $[N]$ such that $g_d = f_n$ and*

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I'} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right| \ll n H_1^{-1/2^r} + \left| \frac{1}{NM} \sum_x \sum_{y \in I'} g_0(x) g_1(x + a_1 y + b_1) \cdots g_d(x + a_d y + b_d) \right|^{1/2^r}, \quad (4.11)$$

for some integers a_i, b_i with a_i distinct, non-zero, and of magnitude at most $H(4H_1)^{rk}$.

Remark. It is important for our purposes that whilst the numbers r and d may depend on the coefficients of the configuration \mathbf{P} , we have upper bounds for these quantities which depend solely on $D(\mathbf{P})$ and n .

Proof. We proceed by induction along the colex order of $\mathbf{m} := D(\mathbf{P})$, proving the inequality (4.11) with absolute constant $8C^2$, where C is the absolute constant occurring in (4.6).

If $k = \max_i \deg P_i = 1$ then we are done on taking $r(\mathbf{m}) = 1$ and $d = n$. Let us therefore assume that $k := \max_i \deg P_i > 1$ and apply Lemma 4.3 to conclude the existence of:

- an interval $I' \subset I$,
- 1-bounded functions $g_0, \dots, g_{n'}$ supported on $[N]$ with $n' \leq 2n$ and $g_{n'} = f_n$,
- polynomials $0, Q_1, \dots, Q_{n'}$ of height at most $H(4H_1)^k$ and distinct non-constant parts,

such that together these satisfy the inequality

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I} f_0(x) f_1(x + P_1(y)) \cdots f_n(x + P_n(y)) \right| \leq CnH_1^{-1/2} + C \left| \frac{1}{NM} \sum_x \sum_{y \in I'} g_0(x) g_1(x + Q_1(y)) \cdots g_{n'}(x + Q_{n'}(y)) \right|^{1/2}. \quad (4.12)$$

Furthermore, writing t for the smallest degree such that $m_t > 0$, we have

$$\mathbf{m}' := D(\mathbf{Q}) = (i_1, \dots, i_{t-1}, m_t - 1, m_{t+1}, \dots) \prec \mathbf{m}$$

for some $i_1 + \dots + i_{t-1} \leq 2n$.

Applying the induction hypothesis, we conclude that there exist positive integers $R' = R(n', \mathbf{m}')$, $r' \leq R'$ and $d \leq 2^{r'} n'$ along with

- an interval $I'' \subset I'$;
- 1-bounded functions $\tilde{g}_0, \dots, \tilde{g}_d$ supported on $[N]$ with $d \leq 2^{r'} n'$ and $\tilde{g}_d = g_{n'} = f_n$;
- integers a_i, b_i with a_1, \dots, a_d distinct, non-zero and of magnitude at most $H(4H_1)^k (4H_1)^{r'k}$.

Moreover, we have the inequality

$$\left| \frac{1}{NM} \sum_x \sum_{y \in I'} g_0(x) g_1(x + Q_1(y)) \cdots g_{n'}(x + Q_{n'}(y)) \right| \leq 8C^2 n' H_1^{-1/2^{r'}} + 8C^2 \left| \frac{1}{NM} \sum_x \sum_{y \in I'} \tilde{g}_0(x) \tilde{g}_1(x + a_1 y + b_1) \cdots \tilde{g}_d(x + a_d y + b_d) \right|^{1/2^{r'}} \quad (4.13)$$

Setting $r = r' + 1$, the observation that $n' \leq 2n$ gives $d \leq 2^r n$ and $|a_i| \leq H(4H)^{rk}$. Furthermore, (4.11) follows with the claimed constant from the inequality

$$\begin{aligned} CnH_1^{-1/2} + C \left(8C^2 n' H_1^{-1/2^{r'}} \right)^{1/2} &\leq (Cn + C(16C^2 n)^{1/2}) H_1^{-1/2^r} \\ &\leq 8C^2 n H_1^{-1/2^r}. \end{aligned}$$

It remains to establish the existence of $R(n, \mathbf{m})$. Define $\mathcal{M}(n, \mathbf{m})$ to be the set

$$\{\mathbf{m}' : m'_1 + \dots + m'_{t-1} \leq 2n, m'_t = m_t - 1, m'_j = m_j \text{ for } j > t\}.$$

Since t is determined by \mathbf{m} , the set $\mathcal{M}(n, \mathbf{m})$ is completely determined by n and \mathbf{m} . By induction along colex, the integer $R(n', \mathbf{m}')$ exists for each

$\mathbf{m}' \in \mathcal{M}(n, \mathbf{m})$ and any valid choice of $n' \leq 2n$, hence the lemma follows on defining

$$R(n, \mathbf{m}) := 1 + \max_{\substack{n' \leq 2n \\ \mathbf{m}' \in \mathcal{M}(n, \mathbf{m})}} R(n', \mathbf{m}').$$

□

Corollary 4.5 (Linearisation for k th power configurations). *Let $f_0, \dots, f_n : \mathbb{Z} \rightarrow [-1, 1]$ be 1-bounded functions supported on $[N]$ and let c_1, \dots, c_n be distinct non-zero integers. Then there exist integers $r = r(n, k)$ and $d = d(n, k)$ such that for any $H \leq N^{1/k}$ there exists $M \leq N^{1/k}$ for which*

$$\left| N^{-\frac{k+1}{k}} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \right| \ll n H^{-1/2^r} + \left| N^{-\frac{k+1}{k}} \sum_x \sum_{y \in [M]} g_0(x) g_1(x + a_1 y + b_1) \cdots g_d(x + a_d y + b_d) \right|^{1/2^r}, \quad (4.14)$$

where g_0, \dots, g_d are 1-bounded functions supported on $[N]$ with $g_d = f_n$, and a_i, b_i are integers with a_i distinct, non-zero, and of magnitude at most $O_c(H^r)$.

Proof. The conclusion follows from Lemma 4.4 and the observation that for this particular configuration \mathbf{P} we have

$$D(\mathbf{P}) = (0, \dots, 0, n, 0, \dots)$$

where the only non-zero entry occurs in the k th place.

To be more precise, the conclusion follows with a height of $O_c(H_1^{kr})$ rather than the claimed $O_c(H_1^r)$. However, we may increase r to kr and the conclusion remains valid. □

5. THE LOCALISED VON NEUMANN THEOREM

In this section we show how a function g_d which has large linear average of the form (4.14) also has large U^d -norm on many short intervals. Recall the definition of the Gowers norm (and its localisation) given in (2.5).

Lemma 5.1 (Linear local von Neumann). *Let $f_0, \dots, f_d : \mathbb{Z} \rightarrow [-1, 1]$ be functions supported on $[N]$ and let a_i, b_i be integers with the a_i distinct and satisfying $0 < |a_i| \leq H$ for all i . Then for any $1 \leq M_1 \leq M$ there exists $M_2 \ll H M_1$ such that we have the inequality*

$$\left| \sum_x \sum_{y \in [M]} f_0(x) f_1(x + a_1 y + b_1) \cdots f_d(x + a_d y + b_d) \right| \ll_d N M_1 + H^2 M \sum_x \frac{\|f_d\|_{U^d(x+[M_2])}}{\|1\|_{U^d(x+[M_2])}}. \quad (5.1)$$

We deduce this from a standard result in which the common difference y is not constrained to lie in a short interval.

Lemma 5.2. *Let $g_0, g_1, \dots, g_d : \mathbb{Z} \rightarrow [-1, 1]$ be functions supported on $[-N, N]$ and let $\mathbf{a}_2, \dots, \mathbf{a}_d \in \mathbb{Z}^2$ be such that $(1, 0), (0, 1), \mathbf{a}_2, \dots, \mathbf{a}_d$ are pairwise linearly independent. Then for $d \geq 2$ we have*

$$\left| \sum_{z_0, z_1} g_0(z_0) g_1(z_1) g_2(\mathbf{a}_2 \cdot \mathbf{z}) \cdots g_d(\mathbf{a}_d \cdot \mathbf{z}) \right| \ll_d N^{2 - \frac{d+1}{2^d}} \|g_d\|_{U^d}.$$

Proof. We proceed by induction on d . For $d = 2$ we use the Fourier transform, as defined in (2.23). If we write (a_0, a_1) for \mathbf{a}_2 , then orthogonality and Hölder's inequality, together with the fact that a_0 and a_1 are both non-zero, gives

$$\begin{aligned} \left| \sum_{\mathbf{z}} g_0(z_0) g_1(z_1) g_2(\mathbf{a}_2 \cdot \mathbf{z}) \right| &= \left| \int_{\mathbb{T}} \hat{g}_0(a_0 \alpha) \hat{g}_1(a_1 \alpha) \overline{\hat{g}_2(\alpha)} d\alpha \right| \\ &\leq \|g_0\|_{L^2} \|g_1\|_{U^2} \|g_2\|_{U^2} \\ &\ll N^{5/4} \|g_2\|_{U^2}. \end{aligned}$$

Here we have used the identity $\|g_i\|_{U^2(\mathbb{Z})} = \|\hat{g}_i\|_{L^4(\mathbb{T})}$.

For the induction step, when $d > 2$, let us again write (a_0, a_1) for \mathbf{a}_2 and

$$G(\mathbf{z}) := g_0(z_0) g_1(z_1) g_3(\mathbf{a}_3 \cdot \mathbf{z}) \cdots g_d(\mathbf{a}_d \cdot \mathbf{z}).$$

Then we have

$$\begin{aligned} \left| \sum_{z_0, z_1} g_0(z_0) g_1(z_1) g_2(\mathbf{a}_2 \cdot \mathbf{z}) \cdots g_d(\mathbf{a}_d \cdot \mathbf{z}) \right| &= \left| \sum_{\mathbf{z}} G(\mathbf{z}) g_2(a_0 z_0 + a_1 z_1) \right| \\ &= \left| \int_{\mathbb{T}} \hat{G}(a_0 \alpha, a_1 \alpha) \overline{\hat{g}_2(\alpha)} d\alpha \right| \\ &\leq \|\hat{g}_2\|_{L^2} \left(\int_{\mathbb{T}} |\hat{G}(a_0 \alpha, a_1 \alpha)|^2 d\alpha \right)^{1/2}. \end{aligned}$$

Interpreting the underlying equations, we see that

$$\begin{aligned} \int_{\mathbb{T}} |\hat{G}(a_0 \alpha, a_1 \alpha)|^2 d\alpha &= \sum_{\mathbf{a}_2 \cdot (\mathbf{z} - \mathbf{w}) = 0} G(\mathbf{z}) G(\mathbf{w}) \\ &= \sum_{\mathbf{h} : \mathbf{a}_2 \cdot \mathbf{h} = 0} \sum_{\mathbf{z}} G(\mathbf{z}) G(\mathbf{z} + \mathbf{h}). \end{aligned}$$

For fixed \mathbf{h} , let us set $\tilde{g}_0 = \Delta_{h_0} g_0$, $\tilde{g}_1 = \Delta_{h_1} g_1$ and $\tilde{g}_i = \Delta_{\mathbf{a}_i \cdot \mathbf{h}} g_i$ for $i \geq 3$. Then applying the induction hypothesis we have

$$\begin{aligned} \sum_{\mathbf{z}} G(\mathbf{z}) G(\mathbf{z} + \mathbf{h}) &= \sum_{\mathbf{z}} \tilde{g}_0(z_0) \tilde{g}_1(z_1) \tilde{g}_3(\mathbf{a}_3 \cdot \mathbf{z}) \cdots \tilde{g}_d(\mathbf{a}_d \cdot \mathbf{z}) \\ &\ll_d N^{2 - d2^{1-d}} \|\Delta_{\mathbf{a}_d \cdot \mathbf{h}} g_d\|_{U^{d-1}}. \end{aligned}$$

Let $c = \text{hcf}(a_0, a_1)$ and set $\mathbf{b} = (b_0, b_1) := c^{-1}(a_1, -a_0) \in \mathbb{Z}^2$. Then we know that the set

$$\{\mathbf{h} \in \mathbb{Z}^2 : \mathbf{a}_2 \cdot \mathbf{h} = 0\}$$

is in bijective correspondence with \mathbb{Z} via the map $h \mapsto h\mathbf{b}$. Since \mathbf{a}_d is not colinear to \mathbf{a}_2 , we have $\mathbf{a}_d \cdot \mathbf{b} \neq 0$, so we are legitimate in the assertion that

$$\begin{aligned} \sum_{\mathbf{h} : \mathbf{a}_2 \cdot \mathbf{h} = 0} \|\Delta_{\mathbf{a}_d \cdot \mathbf{h}} g_d\|_{U^{d-1}} &= \sum_h \|\Delta_{h\mathbf{a}_d \cdot \mathbf{b}} g_d\|_{U^{d-1}} \\ &\leq \sum_h \|\Delta_h g_d\|_{U^{d-1}}. \end{aligned}$$

Given that g_d is supported on $[-N, N]$, the set of integers h for which $\Delta_h g_d$ is not identically zero is contained in the interval $[-2N, 2N]$. Hence by Hölder's inequality

$$\begin{aligned} \left(\sum_h \|\Delta_h g_d\|_{U^{d-1}} \right)^{2^{d-1}} &\leq (4N+1)^{2^{d-1}-1} \sum_h \|\Delta_h g_d\|_{U^{d-1}}^{2^{d-1}} \\ &= (4N+1)^{2^{d-1}-1} \|g_d\|_{U^d}^{2^d}. \end{aligned}$$

Thus

$$\left| \sum_{z_0, z_1} g_0(z_0) g_1(z_1) g_2(\mathbf{a}_2 \cdot \mathbf{z}) \cdots g_d(\mathbf{a}_d \cdot \mathbf{z}) \right| \ll_d N^{\frac{1}{2}+1-d2^{-d}+\frac{1}{2}-2^{-d}} \|g_d\|_{U^d}.$$

□

Proof of Lemma 5.1.

Given $g : \mathbb{Z} \rightarrow [-1, 1]$ one can check that for any $1 \leq M_1 \leq M$ we have

$$\begin{aligned} \sum_{y \in [M]} g(y) &= \frac{1}{M_1^2} \sum_{z_0, z_1 \in [M_1]} \sum_{y \in [M] + z_0 - z_1} g(y - z_0 + z_1) \\ &= \frac{1}{M_1^2} \sum_{z_0, z_1 \in [M_1]} \sum_{y \in [M]} g(y - z_0 + z_1) + O(M_1). \end{aligned}$$

Applying this to the left-hand side of (5.1) and maximising over $y \in [M]$, we deduce that

$$\begin{aligned} \left| \sum_x \sum_{y \in [M]} f_0(x) f_1(x + a_1 y + b_1) \cdots f_d(x + a_d y + b_d) \right| &\ll NM_1 + \\ \frac{M}{M_1^2} \left| \sum_x \sum_{z_0, z_1 \in [M_1]} f_0(x) \tilde{f}_1(x + a_1(z_1 - z_0)) \cdots \tilde{f}_d(x + a_d(z_1 - z_0)) \right|. \end{aligned}$$

Shifting the x variable by $a_1 z_0$ and setting $\mathbf{a}_i := (a_1 - a_i, a_i)$ gives

$$\begin{aligned} \sum_x \sum_{z_0, z_1 \in [M_1]} f_0(x) \tilde{f}_1(x + a_1(z_1 - z_0)) \cdots \tilde{f}_d(x + a_d(z_1 - z_0)) &= \\ \sum_x \sum_{z_0, z_1 \in [M_1]} f_0(x + a_1 z_0) \tilde{f}_1(x + a_1 z_1) \tilde{f}_2(x + \mathbf{a}_2 \cdot \mathbf{z}) \cdots \tilde{f}_d(x + \mathbf{a}_d \cdot \mathbf{z}). \end{aligned}$$

For fixed x , write

$$g_0(z) := f_0(x + a_1 z) 1_{[M_1]}(z), \quad g_1(z) := \tilde{f}_1(x + a_1 z) 1_{[M_1]}(z)$$

and for $i \geq 2$ set

$$g_i(z) := \tilde{f}_i(x+z)1_I(z) \quad \text{where} \quad I := [-3HM_1, 3HM_1].$$

Then our height estimate $|a_i| \leq H$ ensures that

$$\begin{aligned} \sum_{z_0, z_1 \in [M_1]} f_0(x + a_1 z_0) \tilde{f}_1(x + a_1 z_1) \tilde{f}_2(x + \mathbf{a}_2 \cdot \mathbf{z}) \cdots \tilde{f}_d(x + \mathbf{a}_d \cdot \mathbf{z}) = \\ \sum_{z_0, z_1} g_0(z_0) g_1(z_1) g_2(\mathbf{a}_2 \cdot \mathbf{z}) \cdots g_d(\mathbf{a}_d \cdot \mathbf{z}). \end{aligned}$$

By Lemma 5.2

$$\sum_{z_0, z_1} g_0(z_0) g_1(z_1) g_2(\mathbf{a}_2 \cdot \mathbf{z}) \cdots g_d(\mathbf{a}_d \cdot \mathbf{z}) \ll_d (HM_1)^{2 - \frac{d+1}{2d}} \|g_d\|_{U^d}.$$

Writing M_2 for the number of integers in I , the result follows on noting the lower bound $\|1\|_{U^d[M_2]}^{2d} \gg_d M_2^{d+1}$ and the shift invariance

$$\sum_x \|\tilde{f}_d\|_{U^d(x+I)} = \sum_x \|f_d\|_{U^d(x+[M_2])}.$$

□

Combining Lemma 5.1 and the linearisation process (Corollary 4.5), we obtain the required generalised von Neumann theorem.

Corollary 5.3 (Local von Neumann theorem). *Let $f_0, \dots, f_n : \mathbb{Z} \rightarrow [-1, 1]$ be 1-bounded functions supported on $[N]$ and let c_1, \dots, c_n be distinct non-zero integers. Then there exist integers $r = r(n, k)$ and $d = d(n, k)$ such that for any $H \leq N^{1/k}$ and any $0 \leq i \leq n$ there exists an integer $M \ll_{\mathbf{c}} H^r N^{1/k}$ satisfying*

$$\begin{aligned} \left| N^{-\frac{k+1}{k}} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \right| \ll_{\mathbf{c}, k} \\ H^{-1/2r} + \left(\frac{H^{2r}}{N} \sum_x \frac{\|f_i\|_{U^d(x+[M])}}{\|1\|_{U^d(x+[M])}} \right)^{1/2r}. \end{aligned}$$

Proof. Without loss of generality, we may assume that $i = n$. This is clear on re-ordering indices if $i > 0$. If $i = 0$ then we perform the change of variables

$$\begin{aligned} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) = \\ \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x - c_n y^k) f_1(x + (c_1 - c_n) y^k) \cdots f_{n-1}(x + (c_{n-1} - c_n) y^k) f_n(x), \end{aligned}$$

noting that the integers $-c_n, c_1 - c_n, \dots, c_{n-1} - c_n$ are distinct and non-zero.

By Corollary 4.5 there exist integers $r = r(n, k)$ and $d = d(n, k)$ such that for any $H \leq N^{1/k}$ there exists $M \leq N^{1/k}$ for which

$$\left| N^{-\frac{k+1}{k}} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \right| \ll_n H^{-1/2^r} + \left| N^{-\frac{k+1}{k}} \sum_x \sum_{y \in [M]} g_0(x) g_1(x + a_1 y + b_1) \cdots g_d(x + a_d y + b_d) \right|^{1/2^r},$$

where g_0, \dots, g_d are 1-bounded functions supported on $[N]$ with $g_d = f_n$, and a_i, b_i are integers with a_i distinct, non-zero, and of magnitude at most $O_c(H^r)$.

Set $M_1 := \min \{ \lfloor N^{1/k}/H \rfloor, M \}$. Then by Lemma 5.1 there exists $M_2 \ll_c H^r M_1$ for which

$$\left| \sum_x \sum_{y \in [M]} g_0(x) f_1(x + a_1 y + b_1) \cdots g_d(x + a_d y + b_d) \right| \ll_{d,c} NM_1 + H^{2r} M \sum_x \frac{\|f_d\|_{U^d(x+[M_2])}}{\|1\|_{U^d(x+[M_2])}}.$$

The result follows on combining these estimates with the bound $M \leq N^{1/k}$. \square

6. MODIFYING GOWERS'S LOCAL INVERSE THEOREM

In the course of re-proving Szemerédi's theorem, Gowers [Gow01] established the following local inverse theorem for the uniformity norm $\|\cdot\|_{U^d}$.

Gowers's inverse theorem. *For $d \geq 1$ there exist constants $C = C(d)$ and $c = c(d) > 0$ such that the following is true. Suppose that $f : \mathbb{Z} \rightarrow [-1, 1]$ satisfies*

$$\|f\|_{U^d[N]} \geq \delta \|1\|_{U^d[N]}.$$

Then one can partition $[N]$ into arithmetic progressions P_i , each of length at least $c\delta^C N^{c\delta^C}$ such that

$$\sum_i \|f\|_{U^1(P_i)} \geq c\delta^C \sum_i \|1\|_{U^1(P_i)}. \quad (6.1)$$

This result implies that a set $A \subset [N]$ lacking $d+1$ elements in arithmetic progression has size bound $|A| \ll N(\log \log N)^{-\kappa}$ for some small positive constant $\kappa = \kappa(d)$. The precise value of $\kappa(d)$ depends very much on the permissible value of $C = C(d)$ in the local inverse theorem, which Gowers explicitly calculates. For polynomial progressions of the form (1.1), the number of iterative steps required in the linearisation process of §4 means that the degree of $d = d(n, k)$ of the Gowers norm controlling this configuration grows inordinately rapidly in n and k ; so much so that explicit estimates of $C(d)$ are not useful for our purpose. As a consequence, our final density bound (1.2) has an inexplicit exponent of $\log \log N$.

Unfortunately we cannot use Gowers's inverse theorem as stated. Our difficulty is that the theorem gives us information about the U^1 -norm of a function localised to arithmetic progressions, yet we require these progressions to take a special form.

Definition (*k*th power progression). Call an arithmetic progression a *k*th power progression if it has common difference equal to a perfect *k*th power.

We require the following modified version of Gowers's inverse theorem.

Theorem 6.1 (Gowers's inverse theorem for *k*th power progressions). *For $d, k \geq 1$ there exist $C = C(d, k)$ and $c = c(d, k) > 0$ such that the following is true. Suppose that $f : \mathbb{Z} \rightarrow [-1, 1]$ satisfies*

$$\|f\|_{U^d[N]} \geq \delta \|1\|_{U^d[N]}.$$

*Then one can partition $[N]$ into *k*th power progressions P_i , each of length at least $c\delta^C N^{\exp(-1/c\delta^C)}$ such that*

$$\sum_i \|f\|_{U^1(P_i)} \geq c\delta^C \sum_i \|1\|_{U^1(P_i)}.$$

The proof of Theorem 6.1 follows from an elementary, albeit lengthy, modification of the argument of Gowers [Gow01, pp.489-585]. The reader is referred to [Gre02, §5] for a detailed exposition of the argument for the U^3 -norm with $k = 2$. Below we sketch the content of the main modification needed in the general case.

Gowers's argument begins by working over the progression $[N]$ (identified with a subset of the group $\mathbb{Z}/N'\mathbb{Z}$ for some prime $N' \ll_d N$) and proceeds by repeatedly passing to (integer) subprogressions, finally obtaining the subprogressions P_i of the conclusion (6.1). This subprogression refinement takes place in [Gow01, §16] (with an additional refinement taking place in [Gow01, Prop. 17.7]), repeatedly employing results from [Gow01, §5 & §7] to obtain the required subprogression. Each stage passes from a progression of common difference m to a progression of common difference mq , where q arises in one of the following three ways.

- (A1) Passage to a shorter segment of the same progression, as in [Gow01, Prop. 17.7], so that $q = 1$.
- (A2) An application of results in [Gow01, §5], all of which ultimately rest on the following consequence of Weyl's inequality: There exists $c_d > 0$ such that for any $\alpha \in \mathbb{T}$ and $Q \geq 1$ we have

$$\min_{1 \leq q \leq Q} \|\alpha q^d\| \ll_d Q^{-c_d}. \quad (6.2)$$

See for example [Gow01, Lem. 16.1].

- (A3) An application of the fact that the Bohr set

$$B(K, \eta) := \{x \in [-N/2, N/2] : \|\alpha x\| \leq \eta \quad (\alpha \in K)\} \quad (6.3)$$

contains an arithmetic progression of length $\gg \eta N^{1/(1+|K|)}$. See [Gow01, Cor. 7.9–7.10, Lem. 13.4, etc].

If m and q are both perfect k th powers then it follows that mq is a perfect k th power. Since this iteration begins with $m = 1$, which is itself a perfect k th power, it suffices to verify that one can take q equal to a perfect k th power in (A2) and (A3).

This easily follows for (A2) by replacing d with dk in (6.2), so that

$$\min_{1 \leq q \leq Q} \|\alpha q^{dk}\| \ll_{d,k} Q^{-c_{d,k}}.$$

Quantitatively, this replaces the absolute constants $c(d), C(d)$ in Gowers's inverse theorem with constants $c(d, k), C(d, k)$.

We replace (A3) with the following.

Lemma 6.2. *There exists an absolute constant $C = C(k)$ such that the Bohr set $B(K, \eta)$ defined in (6.3) contains a k th power progression of length at least*

$$\gg_k \eta N^{\exp(-C|K|)}.$$

In the proof of Gowers's inverse theorem appearing in [Gow01], all applications of (A3) have $\eta^{-1} \leq C\delta^{-C}$ and $|K| \leq C\delta^{-C}$ for some absolute constant $C = C(d)$. This leads to the passage from an arithmetic progression of length N to a progression of length $c\delta^C N^{c\delta^C}$. Replacing (A3) with Lemma 6.2 therefore passes to an arithmetic progression of length at least

$$c\delta^C N^{\exp(-1/c\delta^C)}.$$

This leads to the final lower bound on the length of k th power progressions obtained in Theorem 6.1.

Lemma 6.2 follows from an application of the following result of Cook [Coo72].

Lemma 6.3 (Simultaneous k th power recurrence). *There exists an absolute constant $C = C(k)$ such that for any $\alpha_1, \dots, \alpha_r \in \mathbb{T}$ and $Q \geq 1$ we have*

$$\min_{1 \leq q \leq Q} \max_{1 \leq i \leq r} \|\alpha_i q^k\| \ll_k Q^{-\exp(-Cr)}. \quad (6.4)$$

Although [Coo72, Theorem 1] has superior dependence on r and k in the exponent of Q , it has an implicit constant in (6.4) depending r . The nature of this dependence is important for our application. We therefore offer the following elementary proof, with implicit constant independent of r , following the arguments of [Gow01, Gre02].

Proof. The $k = 1$ case follows from Kronecker's theorem on simultaneous Diophantine approximation. Let us therefore suppose that $k \geq 2$.

By a weak version of [Woo12, Theorem 1.7] (and [Hei48] for $k = 2$), there is a constant C_k such that for any $\alpha \in \mathbb{T}$ and $Q \geq 1$

$$\min_{1 \leq q \leq Q} \|\alpha q^k\| \leq C_k Q^{-1/k^3}. \quad (6.5)$$

Iteratively applying (6.5), for each $1 \leq i \leq r$ we can find

$$1 \leq q_i \leq Q^{\frac{k^4}{(k^4+1)^i}} \quad \text{with} \quad \|\alpha_i q_1^k \cdots q_i^k\| \leq C_k Q^{-\frac{k}{(k^4+1)^i}}.$$

Setting $q := q_1 \cdots q_r$ we have

$$q \leq Q^{1-(k^4+1)^{-r}} \leq Q$$

and for each i

$$\begin{aligned} \|\alpha_i q^k\| &\leq \|\alpha_i q_1^k \cdots q_i^k\| q_{i+1}^k \cdots q_r^k \\ &\leq C_k Q^k \left(-\frac{1}{(k^4+1)^i} + \frac{k^4}{(k^4+1)^{i+1}} + \cdots + \frac{k^4}{(k^4+1)^r} \right) \\ &= C_k Q^{-k(k^4+1)^{-r}} \\ &\leq C_k Q^{-k^{-5r}}. \end{aligned}$$

The last estimate following from the inequality $k^4 + 1 \leq k^5$ when $k \geq 2$. The lemma follows with $C = 5 \log k$. \square

Proof of Lemma 6.2. Let $r := |K|$. Employing Lemma 6.3, there exists $1 \leq q \leq Q$ such that for any $\alpha \in K$ we have

$$\|\alpha q^k\| \leq C Q^{-\exp(-Cr)}.$$

Let L be the largest non-negative integer satisfying $LQ < N/2$ and $LCQ^{-\exp(-Cr)} < \eta$. Then $B(K, \eta)$ (as defined in (6.3)) contains the progression

$$\{-Lq^k, \dots, -q^k, 0, q^k, \dots, Lq^k\},$$

which has length

$$\begin{aligned} 2L + 1 &\geq \min \{N/(2Q), (\eta/C)Q^{\exp(-Cr)}\} \\ &\gg_k \min \{N/Q, \eta Q^{\exp(-Cr)}\}. \end{aligned}$$

Taking

$$Q := (N/\eta)^{\frac{1}{1+\exp(-Cr)}},$$

so that $N/Q = \eta Q^{\exp(-Cr)}$, we deduce that $2L + 1 \gg_k \eta N^{\exp(-Cr)}$ (after increasing C a little). \square

7. THE DENSITY INCREMENT AND FINAL ITERATION

In this section we combine all of our previous work to establish the following density increment lemma.

Lemma 7.1 (Density increment lemma). *There exist absolute constants $C = C(n, k)$ and $c = c(\mathbf{c}, k) > 0$ such that the following is true. Let A be a subset of $[N]$ of size δN which lacks configurations of the form*

$$x, x + c_1 y^k, \dots, x + c_n y^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\}. \quad (7.1)$$

Suppose that

$$N \geq \exp \exp(1/(c\delta^C)). \quad (7.2)$$

Then there exists a k th power progression P such that

$$|P| \geq N^{\exp(-1/c\delta^C)} \quad \text{and} \quad |A \cap P| \geq (\delta + c\delta^C)|P|.$$

Proof. Throughout the following proof we write $c = c(\mathbf{c}, k) > 0$ and $C = C(n, k)$ for absolute constants dependent only on their respective parameters. Different occurrences of c and C may denote different absolute constants.

Since A lacks configurations of the form (7.1), we have

$$\sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_A(x) 1_A(x + c_1 y^k) \cdots 1_A(x + c_n y^k) = 0.$$

Recall the definition of the balanced function

$$f_A := 1_A - \delta 1_{[N]}.$$

Making the substitution $1_A = \delta 1_{[N]} + f_A$ and expanding, we deduce that there exist f_0, f_1, \dots, f_n satisfying

$$\{f_A\} \subset \{f_0, f_1, \dots, f_n\} \subset \{f_A, \delta 1_{[N]}\}$$

and

$$(2^{n+1} - 1) \left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \right| \geq \delta^{n+1} \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_{[N]}(x) 1_{[N]}(x + c_1 y^k) \cdots 1_{[N]}(x + c_n y^k). \quad (7.3)$$

Notice that (7.2) implies the much weaker estimate

$$N \geq 1/(c\delta^C). \quad (7.4)$$

Taking $c = c(\mathbf{c}, k)$ in (7.4) sufficiently small, our assumption ensures that

$$\sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} 1_{[N]}(x) 1_{[N]}(x + c_1 y^k) \cdots 1_{[N]}(x + c_n y^k) \geq cN^{1+\frac{1}{k}}$$

Incorporating this into (7.3), we deduce that

$$\left| \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{N}} f_0(x) f_1(x + c_1 y^k) \cdots f_n(x + c_n y^k) \right| \geq c\delta^{n+1} N^{1+\frac{1}{k}}. \quad (7.5)$$

Applying the local von Neumann theorem (Corollary 5.3), there exist integers $r = r(n, k)$ and $d = d(n, k)$ such that for any $H \leq N^{1/k}$ there is an integer $M \ll_{\mathbf{c}} H^r N^{1/k}$ satisfying

$$H^{-1/2^r} + \left(\frac{H^{2r}}{N} \sum_x \frac{\|f_A\|_{U^d(x+[M])}}{\|1\|_{U^d(x+[M])}} \right)^{1/2^r} \geq c\delta^{n+1}.$$

Taking $H = C(\mathbf{c}, k)\delta^{-C(n,k)}$ sufficiently large allows us to conclude that

$$\left(\frac{H^{2r}}{N} \sum_x \frac{\|f_A\|_{U^d(x+[M])}}{\|1\|_{U^d(x+[M])}} \right)^{1/2r} \geq c\delta^{n+1}. \quad (7.6)$$

In order for such a choice of H to be permissible, we require that

$$N^{1/k} \geq C(\mathbf{c}, k)\delta^{-C(n,k)},$$

which certainly follows from (7.4) on taking C and c sufficiently large and small (respectively). This choice of H incorporated into (7.6) then gives

$$\sum_x \frac{\|f_A\|_{U^d(x+[M])}}{\|1\|_{U^d(x+[M])}} \geq c\delta^C N.$$

The function $f_A 1_{x+[M]}$ is identically zero unless $x \in [N] - [M]$. Since $M \ll_{\mathbf{c}} H^r N^{1/k}$ we may ensure that $M \leq N$ from (7.4) and the fact that $k \geq 2$ (if $k = 1$ the density increment lemma is proved in [Gow01]). Hence

$$\sum_{|x| < N} \frac{\|f_A\|_{U^d(x+[M])}}{\|1\|_{U^d(x+[M])}} \geq c\delta^C N.$$

It follows that there exists a set $X \subset (-N, N)$ of size $|X| \geq c\delta^C N$ such that for every $x \in X$ we have

$$\|f_A\|_{U^d(x+[M])} \geq c\delta^C \|1\|_{U^d(x+[M])}.$$

Applying Theorem 6.1, we see that for each $x \in X$, there exists a partition of $x + [M]$ into k th power arithmetic progressions $P_{x,i}$ ($i \in I(x)$), each of length at least $c\delta^C N^{\exp(-1/c\delta^C)}$ and such that

$$\sum_{i \in I(x)} \|f_A\|_{U^1(P_{x,i})} \geq c\delta^C \sum_i \|1\|_{U^1(P_{x,i})} = c\delta^C M.$$

Taking the trivial partition $P_{x,1} := x + [M]$ for $x \notin X$, we conclude that

$$\sum_{|x| < N} \sum_{i \in I(x)} \left| \sum_{y \in P_{x,i}} f_A(y) \right| \geq c\delta^C NM. \quad (7.7)$$

Since $f = 1_A - \delta 1_{[N]}$ has mean zero,

$$\begin{aligned} \sum_{|x| < N} \sum_{i \in I(x)} \sum_{y \in P_{x,i}} f_A(y) &= \sum_x \sum_{y \in x+[M]} f(y) \\ &= \sum_{z \in [M]} \sum_x f(x+z) \\ &= 0. \end{aligned}$$

Adding the above to (7.7) we find that

$$\sum_{|x| < N} \sum_{i \in I(x)} \max \left\{ 2 \sum_{y \in P_{x,i}} f(y), 0 \right\} \geq c\delta^C \sum_{|x| < N} \sum_{i \in I(x)} |P_{x,i}|.$$

Hence there exists a k th power arithmetic progression P of length at least $c\delta^C N^{\exp(-1/c\delta^C)}$ such that

$$\sum_{y \in P} f(y) \geq c\delta^C |P|.$$

Finally, our assumption (7.2) ensures that for some $c' \gg c$ we have

$$c\delta^C N^{\exp(-1/c\delta^C)} \geq N^{\exp(-1/c'\delta^C)}.$$

□

The proof of our main theorem quickly follows.

Proof of Theorem 1.1. Suppose that $A \subset [N]$ with $|A| = \delta N$ lacks a configuration of the form

$$x, x + c_1 y^k, \dots, x + c_n y^k \quad \text{with} \quad y \in \mathbb{Z} \setminus \{0\}. \quad (7.8)$$

Then by Lemma 7.1, provided that

$$N \geq \exp \exp(1/c\delta^C),$$

there exists a k th power progression $P = a + q^k \cdot [N_1]$ of length at least

$$N^{\exp(-1/c\delta^C)}$$

such that

$$|A \cap P| \geq (\delta + c\delta^C)|P|.$$

Let $A_1 := \{x \in \mathbb{Z} : a + q^k x \in A \cap P\}$. Then we have obtained a set $A_1 \subset [N_1]$ lacking configurations of the form (7.8) and of density $\delta_1 := |A_1|/N_1$ satisfying $\delta_1 \geq \delta + c\delta^C$.

Setting

$$\delta_0 := \delta, \quad N_0 := N, \quad A_0 := A,$$

let us iteratively apply Lemma 7.1. Provided that

$$N_i \geq \exp \exp(1/c\delta_i^C) \quad (0 \leq i < n), \quad (7.9)$$

there exists a set $A_n \subset [N_n]$ lacking configurations of the form (7.8) and of density $\delta_n := |A_n|/N_n$ satisfying

$$\delta_n \geq \delta_{n-1} + c\delta_{n-1}^C. \quad (7.10)$$

Moreover, we have the length lower bound

$$N_n \geq N_{n-1}^{\exp(-1/c\delta_{n-1}^C)}. \quad (7.11)$$

Using (7.10) gives $\delta_n \geq \delta + nc\delta^C$. Hence if $n \geq 1/(c\delta^C)$ we obtain the contradiction $\delta_n > 1$. It follows that (7.9) cannot hold for $n \geq 1/(c\delta^C)$, so there exists $i \leq 1/(c\delta^C)$ such that

$$N_i < \exp \exp(1/c\delta_i^C) \leq \exp \exp(1/c\delta^C). \quad (7.12)$$

By (7.11), we have

$$N_i \geq N^{\exp(-i/c\delta^C)} \geq N^{\exp(-1/(c\delta^C)^2)}.$$

Altering our values of c and C appropriately, we deduce that

$$\exp \exp(1/c\delta^C) \geq N^{\exp(-1/c\delta^C)}.$$

Taking logarithms twice then gives

$$2/(c\delta^C) \geq \log \log N.$$

This yields (1.2) on re-arranging. \square

REFERENCES

- [BL96] V. Bergelson and A. Leibman, *Polynomial extensions of van der Waerden's and Szemerédi's theorems*, J. Amer. Math. Soc. **9** (1996), no. 3, 725–753. [↑1](#), [↑11](#)
- [Blo14] T. F. Bloom, *A quantitative improvement for Roth's theorem on arithmetic progressions*, preprint available at <http://arxiv.org/abs/1405.5800>. [↑2](#)
- [Coo72] R. J. Cook, *On the fractional parts of a set of points*, Mathematika **19** (1972), 63–68. [↑27](#)
- [Gow98] W. T. Gowers, *Fourier analysis and Szemerédi's theorem*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. (1998), Extra Vol. I, 617–629. [↑1](#)
- [Gow00] ———, *Arithmetic progressions in sparse sets*, Current developments in mathematics 2000, 149–196. [↑1](#)
- [Gow01] ———, *A new proof of Szemerédi's theorem*, Geom. Funct. Anal. **11** (2001), 465–588. [↑1](#), [↑2](#), [↑3](#), [↑4](#), [↑5](#), [↑6](#), [↑25](#), [↑26](#), [↑27](#), [↑30](#)
- [Gre02] B. J. Green, *On arithmetic structures in dense sets of integers*, Duke Math. J. **114** (2002), 215–238. [↑3](#), [↑4](#), [↑6](#), [↑26](#), [↑27](#)
- [GT08] B. J. Green and T. Tao, *The primes contain arbitrarily long arithmetic progressions*. Ann. of Math. **167** (2008), 481–547. [↑5](#)
- [GT09] ———, *New bounds for Szemerédi's theorem II. A new bound for $r_4(N)$* . In *Analytic number theory*, 180–204. Cambridge Univ. Press, 2009. [↑2](#)
- [Hei48] H. Heilbronn, *On the distribution of the sequence $n^2\theta \pmod{1}$* . Quart. J. Math., (1948). 249–256. [↑27](#)
- [Luc06] J. Lucier, *Intersective sets given by a polynomial*, Acta Arith. **123** (2006), no. 1, 57–95. [↑2](#)
- [LR15] N. Lyall and A. Rice, *Difference sets and polynomials*, preprint available at <http://arxiv.org/abs/1504.04904>. [↑3](#)
- [Rot53] K. F. Roth, *On certain sets of integers*, J. London Math. Soc. **28** (1953). 104–109. [↑2](#), [↑4](#)
- [Sá78a] A. Sárközy, *On difference sets of sequences of integers I*, Acta Math. Acad. Sci. Hungar. **31** (1978), 125–149. [↑2](#)
- [Sá78b] ———, *On difference sets of sequences of integers III*, Acta Math. Acad. Sci. Hungar. **31** (1978), 355–386. [↑2](#)
- [Tao13] T. Tao, *A Fourier-free proof of the Furstenberg–Sárközy theorem*, blog post available at <https://goo.gl/CdBd38>. [↑7](#)
- [TZ08] T. Tao and T. Ziegler, *The primes contain arbitrarily long polynomial progressions*, Acta Mathematica **201** (2008), 213–305. [↑6](#)
- [TZ15] ———, personal communication. [↑6](#)
- [TZ16] ———, *Concatenation theorems for anti-Gowers-uniform functions and Host–Kra characteristic factors*, preprint available at <http://arxiv.org/abs/1603.07815>. [↑6](#)

- [Wal00] M. Walters, *Combinatorial proofs of the polynomial van der Waerden theorem and the polynomial Hales–Jewett theorem*, J. London Math. Soc. **61** (2000), 1–12. [↑2](#)
- [Woo12] T. D. Wooley, *Vinogradov’s mean value theorem via efficient congruencing*. Ann. of Math. **175** (2012), no. 3, 1575–1627. [↑27](#)

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